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On the analytic inversion of functions, solution of transcendental equations and infinite self-mappings

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Abstract
The solution of seemingly simple transcendental equations is in effect equivalent to the general problem of analytical inversion of functions. Within a powerful and systematic method, based on the solution of an associated Riemann–Hilbert boundary value problem, beautiful explicit results for various inverse functions of physical importance have been found which inevitably take on the guise of integral representations of these functions. In an attempt to reduce one such solution to a standard-function expression which would then be easy to evaluate, we recognize an infinite ladder self-mapping solution. This new perspective, born out of complicated complex analysis, is straightforwardly and uniquely related to the systematic generation of fast converging expansions within the corresponding regions of single-valuedness of the inverse function.

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1. Introduction

A wide variety of physical problems reduce to the solution $y(x)$ of transcendental equations which can very generally be cast into the form $x = yg(y)$ with $g(y)$ a transcendental function. Although in any particular case it is a more or less routine matter to set up a computer program to determine x for a given value of y, it is very often desirable to develop an approximate analytical representation of the solution, especially if there is an additional dependence on a parameter or if an asymptotic solution is required. Of the numerous ways of constructing such approximations, two are most often used and are therefore worthy of particular consideration, the fixed-point (functional) iteration and Lagrange's reversion formula.

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On the other hand, powerful analytical tools, centred on the solution to the Riemann–Hilbert boundary-value problems [1, 2] have been used ingeniously to generate *exact* analytical solutions to some of the physically interesting transcendental equations. In the early 1970s, Burniston and Siewert [3] developed a scheme to solve a class of transcendental equations, based on the solution of a skilfully formulated associated Riemann–Hilbert problem. Equations that have been treated by this method typically arise as implicit dispersion equations due to the imposition of boundary conditions for the relevant wave equations. The method has been extended [4] and critically assessed [5] by Ioakimidis and Anastasselou who have actually proposed an alternative powerful method that utilizes only the solution of a simple discontinuity problem instead of a Riemann–Hilbert problem. Very recently, Paul and Nkemzi [6] have improved on an earlier solution by Siewert [7] of a problem of considerable physical interest, that of the energy levels of a finite square-well potential. The equation that is of interest in this problem is precisely of the same genesis as the earlier dispersion equations considered by this method; it arises by imposing the usual quantum-mechanical conditions of continuity of the logarithmic derivative of the wavefunction at the boundaries of the well. The relevance of both the explicit and asymptotic expressions for the energy levels as functions of the well's parameters for dynamic phenomena such as revivals and super-revivals of wave packets excited in the well has been emphasized by Aronstein and Stroud [8].

While it is not our purpose to make a full list of the important physical applications of the Riemann–Hilbert method, there is little doubt that some of the ideas exposed below will be applicable beyond the case of the particular equation we are considering. The unifying viewpoint is the treatment of complicated closed-form solutions as the analytical inversions of the corresponding functions. We shall see how, starting from a seemingly intractable solution which requires the computation of a complicated real integral, considerable simplification may be achieved and new insights ensue that appear to be equivalent to a new approach to finite and accurate approximations, not necessarily of a power-series type, for roots of transcendental equations or for inverse functions, more generally.

2. The equation and its solution by the Riemann–Hilbert boundary-value techniques

The function whose inversion we examine is

$$y(x) = x\,\mathrm{e}^{x} \tag{1}$$

with y and x real. If we define

$$\Phi(x, y) \equiv y - x\,\mathrm{e}^{x} \tag{2}$$

then the proper root of the equation $\Phi(x, y) = 0$ defines an implicit function $x = x(y)$, subject to the usual requirements of single-valuedness.

In a substantial generalization of the Siewert–Burniston method, Anastasselou and Ioakimidis have found two equivalent solutions, $x(y)$, within the domain $-\frac{1}{\mathrm{e}} < y < 0$ [4] (figure 1). A survey of their assumptions reveals that in fact these solutions are valid for $-\frac{1}{\mathrm{e}} < y < \mathrm{e}$, whereby $|x| < 1$. Their solutions are no less than *equivalent integral representations* of the inverse function $x = x(y)$ over the latter interval in y, where the inverse is single-valued.

As with all other applications of this method, these are not the only representations. To understand the source of this 'degree of freedom', let us briefly summarize the essence of the problem (see [2] for a detailed exposition). At that, we shall only review the homogeneous Riemann problem, since this particular one has been used for the solution we take up below.

Assume that L is a simple smooth closed contour, dividing the complex plane into interior domain Ω^{+} and exterior domain Ω^{-}. Let the function $D(t)$, nonvanishing and satisfying

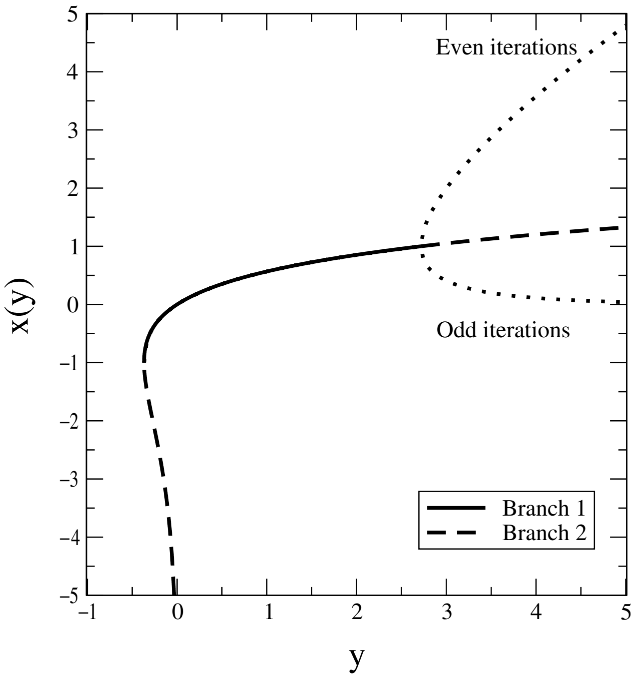

Figure 1. The inverse function $x(y)$. The solid line corresponds to the exponential-ladder solution (branch 1). The dashed lines encompass the two logarithmic-ladder solutions. At $y = \mathrm{e}$, the exponential ladder generates a bifurcation in its corresponding iterative map. Beyond the critical point the logarithmic ladder provides for a convergent self-mapping $f_>$. The lower endpoint of branch 1 exhibits a similar crossover to a stable logarithmic mapping $f_<$.

Hölder's condition, be defined on the contour. The homogeneous Riemann problem is then the problem of finding two functions (or a single piecewise analytic function) $\Phi^+(z)$, analytic in Ω^+, and $\Phi^-(z)$, analytic in Ω^- including $z = \infty$, so that the following linear relation is satisfied on the contour L:

$$\Phi^+(t) = D(t)\Phi^-(t). \tag{3}$$

The function $D(t)$ is the coefficient of the Riemann problem. The index ν of this function plays a prominent part in the solution. It is defined as

$$\nu = \operatorname{Ind} D(t) = \frac{1}{2\pi}[\arg D(t)]_L. \tag{4}$$

Under the conditions imposed on $D(t)$, its index on a closed contour is a non-negative integer, in which case the homogeneous Riemann problem has $\nu + 1$ linearly independent solutions, labelled by $k = 0, 1, 2, \ldots, \nu$:

$$\Phi^+(z) = z^k \exp G^+(z) \tag{5}$$

$$\Phi^-(z) = z^{k-\nu} \exp G^-(z). \tag{6}$$

The function $G(z)$ is calculated from the Cauchy-type integral:

$$G(z) = \frac{1}{2\pi\mathrm{i}} \int_L \mathrm{d}\tau \frac{\ln[\tau^{-\nu} D(\tau)]}{\tau - z}. \tag{7}$$

The general solution is then suitably represented as

$$\Phi^{+}(z) = P_\nu(z)\exp(G^{+}(z)) \tag{8}$$

$$\Phi^{-}(z) = z^{-\nu}P_\nu(z)\exp(G^{-}(z)) \tag{9}$$

where $P(z)$ is a polynomial of degree ν, and contains $\nu + 1$ arbitrary constants. The homogeneous problem is unsolvable for negative index ν.

In the adaptation of the method and with the contour used in [4], $\nu = 0$, hence, there is just one arbitrary constant. This is quite enough for the generation of a one-dimensional continuum of equivalent analytical solutions. Anastasselou and Ioakimidis have exhibited two such solutions and we take up one of them for further discussion [4][4]:

$$x = y\exp\left\{-\frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta\,\log[R^2 + I^2]\right\} \tag{10}$$

where R and I are the following functions of both y and θ:

$$R(\theta, y) = \exp(\cos(\theta))\cos(\sin(\theta)) - y\cos(\theta) \tag{11}$$

$$I(\theta, y) = \exp(\cos(\theta))\sin(\sin(\theta)) + y\sin(\theta) \tag{12}$$

while $|x| < 1$ and $-1/\mathrm{e} < y < \mathrm{e}$. Accordingly,

$$R^2 + I^2 = \exp(2\cos(\theta))\rho^2(\theta, y) \tag{13}$$

and, after taking the logarithm as in equation (10) and integrating over θ, one remains with

$$x(y) = y\exp\left\{\frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta(-\log(\rho^2))\right\} \tag{14}$$

where

$$\rho(\theta, y) = \sqrt{1 - 2\xi z + z^2} \tag{15}$$

with

$$\xi \equiv \cos(\theta + \sin\theta) \tag{16}$$

$$z \equiv y\,\mathrm{e}^{-\cos\theta} \tag{17}$$

$$|z| < 1. \tag{18}$$

The form of equation (14) is suggestively chosen to point out that the integrand can actually be viewed as the generating function of the Gegenbauer ultraspherical polynomials $C_m^{(0)}(\xi)$, so that [9]

$$-\log(\rho^2) = \sum_{k=1}^{\infty} C_k^{(0)}(\xi)z^k \qquad (|z| < 1). \tag{19}$$

Changing the order of summation and integration in the exponential of $x = x(y)$ leads one to consider the representation

$$x(y) = y\exp\sum_{k=1}^{\infty}\left(A_k y^k\right) \tag{20}$$

[4] Incidentally, there is a misprint in equation (4.5b) of [4], where the sign preceding the second term must be a plus. This bears no consequence to any of the results reported further on in that paper.

with the numbers A_k given by

$$A_k = \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\theta\, \mathrm{e}^{-k\cos\theta} C_k^{(0)}(\xi). \tag{21}$$

At this point, one observes that the Gegenbauer polynomials that appear in the above are simply related to the Chebyshev polynomials of the first kind $T_k(x)$ [9]:

$$C_k^{(0)}(\cos\Phi) = \frac{2}{k} T_k(\cos\Phi) = \frac{2}{k}\cos(k\Phi) \tag{22}$$

where $\Phi \equiv \theta + \sin\theta$. In the last step, we have used the identity $T_k(\cos u) = \cos(k\cos u)$. One thus needs to evaluate the integrals B_k $(k \geqslant 1)$:

$$B_k = \int_0^{\pi} \mathrm{d}\theta\, \mathrm{e}^{-k\cos\theta} \cos k(\theta + \sin\theta). \tag{23}$$

These are easily transformed to

$$B_k = (-1)^k \int_0^{\pi} \mathrm{d}\theta\, \mathrm{e}^{+k\cos\theta} \cos k(\theta - \sin\theta) \tag{24}$$

and evaluated to [10]

$$B_k = (-1)^k \pi \frac{k^2}{k!}. \tag{25}$$

Inserting back into the exponential in equation (20), one runs into the series expansion of another exponential, namely, $\exp(-y)$, so that it seems as if we have dramatically reduced the initial integral representation of the inverse function $x(y)$ to a two-step exponential:

$$x(y) = y\,\mathrm{e}^{-y\,\mathrm{e}^{-y}} \qquad (-1/\mathrm{e} < y < \mathrm{e}). \tag{26}$$

The first thing one realizes about the compact solution in equation (26) is that, unfortunately and in contrast to the integral-representation solution of [4], it does not represent the exact analytic inversion. To see this, one need only insert $x(y)$ into the starting equation $y = x\exp(x)$. On the other hand, a straightforward power-series expansion about zero indicates that the inversion found departs from the exact one in as high as the fourth order in x. We denote as 'exact' the expansion for the inverse function as prescribed by the Lagrangian formula [11, 12][5]: if $y = f(x)$, $y_0 = f(x_0)$, $f'(x_0) \neq 0$, then

$$x = x_0 + \sum_{k=1}^{\infty} \frac{(y - y_0)^k}{k!} \left[\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} \left(\frac{x - x_0}{f(x) - y_0} \right)^k \right]_{|x=x_0} \tag{27}$$

where $f(x) = x\,\mathrm{e}^x$, $f'(x) = (x+1)\,\mathrm{e}^x$, $x_0 = 0$ and $y_0 = 0$.

The failure of the above procedure to recover a simple expression for the inverse function, starting from the exact integral-representation solution, lies in the logarithmic singularity of the integrand in equation (14) at $\theta = \pi$ which ruins the uniform convergence of the expansion in terms of Gegenbauer polynomials and thus invalidates the change of the order of integration and summation. One lesson is that almost certainly there is no way to represent the inverse function in terms of standard functions. Another lesson is to take a closer look at the 'solution' reached and understand why it is so successful in providing a higher order accuracy about a particular point $(x = y = 0)$.

[5] In [12] one can find the original, more general theorem of Lagrange and the conditions of its validity.

3. The identification of a ladder solution

One recognizes by inspection that the solution exhibited in equation (26) may be extended to include further steps or levels of the exponential and that, in fact, the infinite ladder thus obtained *is* the exact inversion which is sought for. To see this, define

$$x(y) = y\,L(y) \tag{28}$$

where the ladder $L(y)$ itself is defined as

$$L(y) = \exp(-y\exp\{-y\exp[-y\exp(\cdots)]\}). \tag{29}$$

A formal insertion into the starting equation to be solved indicates that one does reach an identity.

It is now not difficult to check that taking only a finite number of steps of the ladder $L(y)$ to make it computable pushes the accuracy of the inverse function one order of magnitude higher, when judged upon by a series expansion about $y = 0$. That is, any further step in the ladder makes the cut-off closed-form analytic solution converge to the exact inversion to one further order of magnitude. The coefficient of the first term of discrepancy has the magnitude of 1 (see the next section for a general clarification of this fact). For instance, take $n = 7$, then the required identity when the inverse function is inserted into the equation $x = y\exp(y)$ is satisfied up to $\mathrm{O}(x^8)$ with a value of 1 for the coefficient of the first departing term.

There are two immediate conclusions at this stage. First, a formal closed-form solution in terms of an infinite ladder of a simple exponential has been derived for the inverse $x(y)$ to the function $y(x) = x\exp(x)$ (with $|x| < 1$, $-1/\mathrm{e} < y < \mathrm{e}$), starting with the integral representation of this same inverse that results from the application of the Riemann–Hilbert boundary-value technique. Second, it has been recognized that cutting the infinite ladder at any given number n of its steps gives a controlled closed-form finite representation, which is not in the form of a power-series and which is correct to the same order n when compared with the 'exact' power-series expansion about $x = y = 0$ as prescribed, e.g., by the method of Lagrange. Thus, from the perspective of analytical inversion, the inverse function $x(y)$ is the limit (infinite ladder) of a series of relatively simple (finite-ladder) functions, featuring a recurring (self-mapping) pattern.

A further perspective is described in the next section.

4. The infinite ladder as an iterative map

The infinite ladder provides a bridge to cross over from the Riemann–Hilbert boundary-value techniques to an iterative understanding of the problem. One proceeds via a simple rearrangement of the form

$$x = y\ \exp(-x) \tag{30}$$

to set up an iterative map with the function to be iterated given by the right-hand side of equation (30), so that taking m successive iterations is precisely equivalent to keeping m steps in the exponential ladder $L(y)$. Moreover, by reinserting the mth order iteration into the initial equation one establishes easily that the identity is satisfied up to $O(x^{m+1})$ and the coefficient of the first deviating term is equal to the mth power of the logarithmic derivative of the right-hand side of equation (30), evaluated at $x = 0$ $(y = 0)$. Since

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log(\exp(-x))] = -1 \tag{31}$$

the coefficient in question is just $(-1)^k$. This has the magnitude of unity as was remarked earlier.

But there is nothing special about the specific function we have considered up to now, so far as an iterative map of the form $f(x) = y/g(x)$ is considered, which corresponds to setting up an inversion scheme for $y = xg(x)$. In fact, the following general statement holds true. Assume that $xg(x)$ vanishes linearly for $x \to 0$. (The property which is actually instrumental is merely that $g'(0) \cdot g(0) \neq 0$.) Set up the iteration map $f(f(\cdots f(y)\cdots))$. (This has the optical appearance of an actual ladder with $g(x) = \exp(x)$ as displayed in the previous section.) Take a finite approximation of order m:

$$f(f(\cdots f(y)\cdots)) \qquad (m \text{ times}) \tag{32}$$

and keep at least $(m+1)$ terms in a power series expansion about $y = 0$. This gives a power series for $x(y)$. Back-substitute the initial function $y = xg(x)$ and expand to the same order m. (The result of this would be an identity of the form $x = x$, if we had the exact inverse function, and not the mth order approximation at our disposal.) It is then true that

$$x = x\left[1 + \left(\frac{\mathrm{d}}{\mathrm{d}x}\log(g(x))_{|x=0}\right)^m x^m\right] + \mathrm{O}(x^{m+2}). \tag{33}$$

The self-mapping method for the solution of nonlinear equations is well known, usually under the name of 'the method of iteration' [13] and the transcendental equations are just one class from among the nonlinear equations. The result of equation (33) is probably not unexpected, but here it is derived very naturally. The rate of convergence it indicates is not at all that fast and is, for instance, much less than that of the Newton–Raphson method which, on the other hand, has its own deficiencies [14]. The implementation of the finite ladder for computational purposes still appears superior to using the original exact integral representation for the inverse function, since it avoids computing any integral at all.

Let us recall at this stage that we have only considered a restricted domain in the variable x ($|x| < 1$) for the inversion of $y(x) = x\exp(x)$ ($-1/\mathrm{e} < y < 1$) (branch 1 in figure 1). A look at the plot of this function tells us that this is only a subdomain of one of two monotonic branches where an inverse function can be uniquely defined. For $x < -1$ and $x > 1$ we have no longer an integral representation of the inversion as a starting point, since the solution provided by [4] is restricted to $|x| < 1$ by construction. Besides, there is no guarantee that, even if one spends time and effort to find such a representation in the spirit of a solution to a suitable Riemann–Hilbert problem, it would then be feasible to transform it to a conclusive form. However, with the insights about the infinite-ladder solution and its relation to iterative maps, we are in a position to discuss the appropriate analytic inversions for these two remaining portions in the variable x.

For the region specified by $x < -1$, $-1/\mathrm{e} < y < 0$, an infinite-ladder solution is easily identified as

$$x(y) = L_<(y) \tag{34}$$

with the ladder $L_<(y)$ now defined as

$$L_<(y) = -\log\left(\frac{\log\frac{\log(\cdots)}{-y}}{-y}\right). \tag{35}$$

The first three steps, i.e. approximations, in this ladder are

$$x_1(y) = \log(-y) \tag{36}$$

$$x_2(y) = \log(-y) - \log(-\log(-y)) \tag{37}$$

$$x_3(y) = \log(-y) - \log[\log(-\log(-y)) - \log(-y)] \tag{38}$$

and further on along this pattern. Note that there is no difficulty in writing down the *m*th term in this series of functions whose $m \to \infty$ limit is the exact formal solution. From the point of view of the corresponding iterative mapping, setting off with a map of the form $f_<(x) = \log\left(\frac{-y}{-x}\right)$, the *m*th approximation takes on the appearance of equation (32),

$$f_<(f_<(\cdots f_<(y)\cdots)) \qquad (m \text{ times}) \tag{39}$$

shown as the 'branch 2' curve in figure 1, corresponding to $-\frac{1}{\mathrm{e}} < y < 0$.

Quite analogously, for the region $x > 1$, $y > \mathrm{e}$ one can check that the formal inversion of our starting equation has pretty much the same appearance as just given:

$$x(y) = L_>(y) \tag{40}$$

with the corresponding infinite ladder defined by

$$L_>(y) = -\log\left(-\frac{-\log\frac{-\log(\cdots)}{y}}{y}\right). \tag{41}$$

The first three functions in the infinite functional series are now

$$x_1(y) = \log(y) \tag{42}$$

$$x_2(y) = \log(y) - \log\log(y) \tag{43}$$

$$x_3(y) = \log(y) - \log[\log(y) - \log\log(y)] \tag{44}$$

and so forth ('branch 2' in figure 1 with $\mathrm{e} < y < \infty$). Here, an iterative mapping can be initiated by using $f_>(x) = \log(y/x)$, where now both x and y are positive (actually, greater than 1 and e, respectively).

Let us note that there is no simple way to compare the finite approximations to $L_<(y)$ and $L_>(y)$ with an 'exact' solution of the Lagrangian power-series type which was at our disposal for the ladder $L(y)$. In fact, for the last case we considered ($L_>(y)$), it has been shown [13] that it takes a considerable additional effort to obtain an asymptotic expansion that has the Lagrangian-type appearance, whereby the coefficients in this expansion whose values would be important for an estimate of the convergence are not at all easy to calculate and come about from an additional integration procedure. While we do not want to go into the details of the arising asymptotic expansions at this place, the formal analogy of the infinite-ladder solutions $L_<(y)(-1/\mathrm{e} < y < 0)$ and $L_>(y)(y > \mathrm{e})$ implies an analogous appearance for the respective asymptotic expansions. Hence, the developments for $L_>(y)$ due to de Bruijn [13] can easily be adapted for the asymptotic expansion of its counterpart $L_<(y)$.

Looking at the plot of the function which was to be inverted, $y(x) = x\exp(x)$, it might appear unexpected that it was necessary to develop three formal ladder solutions, given that the function has only *two* distinct regions of monotonicity to the left and right of $x = -1$, respectively. One expects to have a unique single-valued inversion over each region of monotonicity. Why then, for $x > -1$, do we need both the ladder $L(y)$ and the ladder $L_>(y)$ which are apparently different? To examine this problem more closely, one needs to make use of the iterative or approximating aspect of the corresponding solutions. Taking high-order iterations of the $L(y)$-ladder, which poses no problem numerically, brings to light a remarkable bifurcation which sets in precisely at $y = \mathrm{e}$ $(x = 1)$. One finds that taking an odd or even number of high-order steps in the ladder causes the approximation to bifurcate. This bifurcation is shown in figure 1 (dotted lines), where a large number of iterations have been effected. The upper dotted curve is illustrative for an iteration with an even number of steps ($n = 10^5$ in this example), while the lower dotted curve is an odd iteration with $n = 10^5 + 1$ steps. The analytical condition, locating this instability, is $\left|\frac{\mathrm{d}}{\mathrm{d}x} y\,\mathrm{e}^{-x}\right| \geqslant 1$ when evaluated at $y = x\,\mathrm{e}^x$, which leads to $|-y\,\mathrm{e}^{-x}|_{|y=x\,\mathrm{e}^x|} = |x\,\mathrm{e}^x\,\mathrm{e}^{-x}| = |x| = 1$ at the bifurcation point. Proceeding to a converging map beyond $x = 1$ (i.e. for $y > \mathrm{e}$) is facilitated by iterating the inverse map to the pre-bifurcating one. This inverse map is $\log(x/y) \equiv f_>$ as above. Conversely, iterations of $f_>$ become unstable to a two-cycle instability when $x < 1$, $y < \mathrm{e}$.

More generally, the integral representations which arise from the proper application of the Riemann–Hilbert boundary problem technique represent beyond any doubt an extremely useful starting point for the systematic development of asymptotic expansions. An excellent, recent and important example is provided by the asymptotic expansion of the energy for the allowed levels in a finite quantum well in terms of the naturally defined dimensionless parameters of the physical problem [6, 8].

5. Discussion

We have used the integral representation resulting from an application of the Riemann–Hilbert boundary value technique as a starting point for finding a simple analytical inversion of the function $y(x) = x\exp(x)$. While a logarithmic singularity in the integral representation is prohibitive to finding a simple expression, its effects have not been destructive enough to prevent us from recognizing a formally exact infinite-ladder solution on the whole real axis (the starting representation was only given for $-1 < x < 0$, i.e. $-1/\mathrm{e} < y < 0$). These solutions can be viewed as the limits of a functional series with a given composition law. Each successive function which is the ladder evaluated at a given finite number of steps provides a systematically improving approximation. The branches match continuously, but also give rise to an additional bifurcation which can be examined numerically. So apart from the face value of the formal exact inversion, the present analysis provides a generalized view of the numerical computation of the root of the transcendental equation on the whole real axis.

For the region where a comparison with the 'exact' Lagrangian-type of expansion is possible, our development allows for a compact approximation which is exact to the order $m + 1$ if the m-step approximating function is used (equation (33)). The exact infinite ladders below $y = -1/\mathrm{e}$ and above $y = \mathrm{e}$ are of logarithmic type. They provide a basis for asymptotic expansions which are not of power-series type. In particular, not only does the 'positive' ($L_>(y)$) ladder provide for the nontrivial expansion in terms of $\log(y)$ and $\log\log(y)$ as known from earlier work by de Bruijn [13], but the formal similarity of the 'negative' ($L_<(y)$) and 'positive' ($L_>(y)$) ladders allows for immediate application of de Bruijn's developments to the region of negative values of the variables. This makes it compellingly obvious that the inversion formulae which typically result from the application of the Riemann–Hilbert technique provide a very useful starting point for the systematic development of asymptotic expansions which, in the context of the method, invariably appear as expansions of an integral representation of the inverse function.

The techniques required for the solution of the Riemann–Hilbert problem are of much wider scope than the present study. They have already brought about some gems of analytic results, such as the inversion of the relevant Bose–Einstein and Fermi–Dirac integrals by Leonard [15] and Nieto [16] which led to closed-form explicit equations of state $P = P(T, V)$ for both the ideal Fermi–Dirac and Bose–Einstein gases for both the nonrelativistic [15] and relativistic [16] cases. Clearly, in these cases the method was a tool to solve the untrivial integral equations for the chemical potential $\mu = \mu(P, T)$ or $\mu = \mu(V, T)$. Interestingly, the above-mentioned authors used the exact unwieldy solutions as starting points for developing some asymptotic approximations. This reiterates our point. In fact, in view of the inherent 'degrees of freedom', related to the existence of $\nu + 1$ arbitrary constants for a problem of non-negative index ν, there appears to be a vast playground for identifying such closed-form solutions from among the infinitely many equivalent ones that would lead to useful and eventually superior asymptotic expansions.

Speaking of the particular equation we have considered here, it is worth noting that the number of physical problems where it arises is astonishingly large and, accordingly, so are the prospective applications of the insights reported here. The early motivation for an exact solution was rooted in the theory of neutron moderation (see [4] and references cited therein) and of population growth [17]. However, the range of possible applications stretches out to recent magnetic domain imaging techniques for ultrathin ferromagnetic films [18]. In duly scaled physical variables, the equation that gives the relation between the (scaled) stripe domain width $\bar{D}$ as a function of the (scaled) thickness $\bar{T}$ of the film has precisely the appearance $\bar{D} = \bar{T}\exp(\bar{T})$. Hence, the inversion $\bar{T} = \bar{T}(\bar{D})$ gives the direct answer to the question of what the thickness has to be so that the stripe domains are of a given width. So far, this micromagnetic problem has not been approached from such a perspective which is, however, in tune with the modern material science aspirations for engineering materials with desired properties. An even more recent example is an exactly soluble model for the film growth in samples with two moving boundaries [19].

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